

CHOW QUOTIENTS OF TORIC VARIETIES AS MODULI OF STABLE LOG MAPS

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ABSTRACT. Let X be a projective normal toric variety and T_0 a rank one subtorus of the defining torus of X . We show that the normalization of the Chow quotient $X//T_0$, in the sense of Kapranov-Sturmfels-Zelevinsky, coarsely represents the moduli space of stable log maps to X with discrete data given by $T_0 \subset X$.

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1. INTRODUCTION

Throughout, we work over an algebraically closed field k of characteristic 0.

Chow quotients of toric varieties were introduced by Kapranov, Sturmfels, and Zelevinsky in [KSZ91]. Given a projective normal toric variety X and a subtorus T_0 of the defining torus T , the *Chow quotient* $X//T_0$ has the property that its normalization is the smallest toric variety which maps onto all GIT quotients of X by T_0 . We show in this paper that when T_0 has rank one, the normalization of $X//T_0$ can be reinterpreted as the coarse moduli space of the stack of stable log maps, introduced by Abramovich and the first author [Che11, AC11], and independently by Gross and Siebert [GS11].

We begin by recalling the construction of $X//T_0$. For every point $x \in X$, the closure $Z_x := \overline{T_0 x}$ of the orbit of x under T_0 is a subvariety of X . For $x \in T$, the orbit closures Z_x have the same dimension and homology class. We therefore obtain a morphism from $T' := T/T_0$ to the Chow variety $C(X)$ of algebraic cycles of the given dimension and homology class. The Chow quotient $X//T_0$ is defined as the closure of T' in $C(X)$. It is a toric variety and the fan of its normalization is given explicitly in [KSZ91, §1].

Further assume now that T_0 is a rank one torus. Let Z_1 be the closure of T_0 in X . Then its normalization \tilde{Z}_1 is isomorphic to \mathbb{P}^1 and the induced morphism

$$f_1 : \mathbb{P}^1 \longrightarrow X$$

can be viewed as a stable map with two marked points $\{0, \infty\} = \mathbb{P}^1 \setminus f_1^{-1}(T)$. Let β_0 be the curve class of the stable map f_1 and let c_0 and c_∞ be the contact orders of 0 and ∞ with respect to the toric boundary $X \setminus T$. Roughly speaking, c_0 and c_∞ are functions which assign to the marked points their orders of tangency with the components of $X \setminus T$ (see [ACGM11] for more details). In the toric case, the contact orders can be explained as the slopes and weights of the unbounded edges of tropical curves associated to stable log maps, see Section 3.3.

Our primary object of study in this paper is the stack $\mathcal{K}_{\Gamma_0}(X)$ parameterizing stable log maps from rational curves with two marked points to X such that the curve class is β_0 and the marked points have contact orders given by c_0 and c_∞ ; here $\Gamma_0 := (0, \beta_0, 2, \{c_0, c_\infty\})$ keeps track of the discrete data consisting of genus, curve class, number of marked points, and their tangency conditions. Our main result is:

Theorem 1.1. *The normalization of $X//T_0$ is the coarse moduli space of $\mathcal{K}_{\Gamma_0}(X)$.*

Remark 1.2. In particular, we see that $\mathcal{K}_{\Gamma_0}(X)$ is irreducible.

In the process of proving Theorem 1.1, we obtain an alternative description of $\mathcal{K}_{\Gamma_0}(X)$ which is more akin to the construction of the Chow quotient. As we saw above, $X//T_0$ is defined as the closure of $T' := T/T_0$ in the Chow variety $C(X)$. Replacing $C(X)$ by other moduli spaces, we obtain alternate spaces birational to $X//T_0$. Letting Z_x be the orbit closure $\overline{T_0 x}$ as above, we see that for all $x \in T$, the normalization \tilde{Z}_x is isomorphic to \mathbb{P}^1 . Thus, we obtain a stable map

$$f_x : \mathbb{P}^1 \longrightarrow X$$

with marked points $\{0, \infty\} = \mathbb{P}^1 \setminus f_x^{-1}(T)$. These f_x all have curve class β_0 , and we obtain an immersion

$$T' \longrightarrow \mathfrak{M}_{0,2}(X, \beta_0),$$

where $\mathfrak{M}_{0,2}(X, \beta_0)$ denotes the Kontsevich space of stable maps to X with genus 0, curve class β_0 , and two marked points. In analogy with the construction of the Chow variety, we let \mathfrak{M} denote the closure of T' in $\mathfrak{M}_{0,2}(X, \beta_0)$. Then we have:

Theorem 1.3. *$\mathcal{K}_{\Gamma_0}(X)$ is the normalization of \mathfrak{M} .*

Remark 1.4. There is an analogous picture if one assumes that X is an affine normal toric variety and replaces $\mathfrak{M}_{0,2}(X, \beta_0)$ above by the toric Hilbert scheme, as defined in [PS02]. That is, for all $x \in T$, the Z_x are T' -invariant closed subschemes of X which have the same discrete invariants. We therefore obtain an immersion from T' to an appropriate toric Hilbert scheme. The closure of T' in this toric Hilbert scheme is called the main component. In [Ols08, Thm 1.7], Olsson shows that the normalization of the main component has a natural moduli interpretation in terms of log geometry. Theorem 1.3 above can therefore be viewed as an analogue of Olsson's theorem, replacing his use of the toric Hilbert scheme by the Kontsevich space. That is, we show that the normalization of \mathfrak{M} carries a moduli interpretation in terms of stable log maps.

Recall that given any collection of discrete data $\Gamma = (g, \beta, n, \{c_i\}_{i=1}^n)$, it is shown in [Che11, AC11, GS11] that there is a proper Deligne-Mumford stack $\mathcal{K}_\Gamma(X)$ which parameterizes stable log maps to X from genus g curves with n marked points having curve class β and contact orders given by the c_i .¹ We show in Proposition 2.1 that if $g = 0$, then $\mathcal{K}_\Gamma(X)$ is log smooth, and in particular normal. This is a key ingredient in the proof of Theorem 1.3, which we give in Section 2. In Section 3, following [NS06, GS11], we explain the relationship between tropical curves and stable log maps to toric varieties. While the use of tropical curves is not strictly necessary for this paper, they serve as a convenient tool to study the boundary of $\mathcal{K}_\Gamma(X)$. Theorem 1.1 is then proved in Section 4.

Prerequisites: We assume the reader is familiar with logarithmic geometry in the sense of Fontaine-Illusie-Kato (see for example [Kat89] or [Ogu06]).

¹Strictly speaking, [Che11, AC11] only consider log schemes which are generalized Deligne-Faltings (see Definition A.1), so to apply their theory, one must first show that the natural log structure on X satisfies this hypothesis. This is done in Proposition A.4, which we relegate to an appendix since the theory developed in [GS11] is already known to apply to toric varieties.

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2. LOG SMOOTHNESS AND IRREDUCIBILITY

Throughout this section, X is a projective normal toric variety of dimension d and Γ is the discrete data $(0, \beta, n, \{c_i\})$. Let T be the defining torus of X and M be the character lattice of T .

Proposition 2.1. $(\mathcal{K}_\Gamma(X), \mathcal{M}_{\mathcal{K}_\Gamma(X)})$ is log smooth over (k, \mathcal{O}_k^*) . Moreover, $\dim \mathcal{K}_\Gamma(X) = \dim X + n - 3$.

Proof. The universal curve on $\mathcal{K}_\Gamma(X)$ induces a morphism of log stacks:

$$\pi : (\mathcal{K}_\Gamma(X), \mathcal{M}_{\mathcal{K}_\Gamma(X)}) \longrightarrow (\mathfrak{M}_{0,n}, \mathcal{M}_{\mathfrak{M}_{0,n}}),$$

where $(\mathfrak{M}_{g,n}, \mathcal{M}_{\mathfrak{M}_{g,n}})$ denotes the log stack of (g, n) -prestable curves; see [Kat00] and [Ols07, Thm 1.10] for the definition and construction of this log stack. Since $(\mathfrak{M}_{g,n}, \mathcal{M}_{\mathfrak{M}_{g,n}})$ is log smooth over (k, \mathcal{O}_k^*) , it suffices to show that π is log smooth. By [Ols03, Thm 4.6], this is equivalent to showing that the induced morphism

$$\pi' : \mathcal{K}_\Gamma(X) \longrightarrow \mathcal{Log}(\mathfrak{M}_{0,n}, \mathcal{M}_{\mathfrak{M}_{0,n}})$$

of stacks is smooth, where $\mathcal{Log}_{(S, \mathcal{M}_S)}$ is the stack of log morphisms to a log scheme (S, \mathcal{M}_S) , as defined in the introduction of (loc. cit.).

Let $i : \text{Spec } A \rightarrow \text{Spec } A'$ be a square zero thickening of Artin local rings and let

$$\begin{array}{ccc} \text{Spec } A & \longrightarrow & \mathcal{K}_\Gamma(X) \\ i \downarrow & & \downarrow \pi' \\ \text{Spec } A' & \longrightarrow & \mathcal{Log}(\mathfrak{M}_{0,n}, \mathcal{M}_{\mathfrak{M}_{0,n}}) \end{array}$$

be a commutative diagram. We may view this as a commutative diagram of log stacks, by endowing the Artin local rings with the log structure pulled back from $\mathcal{Log}(\mathfrak{M}_{0,n}, \mathcal{M}_{\mathfrak{M}_{0,n}})$. Hence the two vertical arrows are strict. Denote the induced log structures on $\text{Spec } A$ and $\text{Spec } A'$ by \mathcal{M}_A and $\mathcal{M}_{A'}$, respectively. We therefore have a log smooth curve h' , a cartesian diagram

$$\begin{array}{ccc} (C, \mathcal{M}_C) & \longrightarrow & (C', \mathcal{M}_{C'}) \\ h \downarrow & & \downarrow h' \\ (\text{Spec } A, \mathcal{M}_A) & \longrightarrow & (\text{Spec } A', \mathcal{M}_{A'}) \end{array}$$

and a minimal stable log map $f : (C, \mathcal{M}_C) \rightarrow (X, \mathcal{M}_X)$, which we must show deforms to a minimal stable log map $f' : (C', \mathcal{M}_{C'}) \rightarrow (X, \mathcal{M}_X)$. Since the minimality condition is open by [Che11, Prop 3.5.2], it suffices to show that f deforms as a morphism of log schemes.

By standard arguments in deformation theory, it is enough to consider the case where the kernel \mathcal{I} of $A' \rightarrow A$ is principal and killed by the maximal ideal \mathfrak{m} of A' . Then the obstruction to deforming f to a morphism of log schemes lies in

$$\text{Ext}^1(f_0^* \Omega_{(X, \mathcal{M}_X)/k}^1, \mathcal{O}_{C_0}) \otimes_k \mathcal{I}$$

where f_0 denotes the reduction of $f \bmod \mathfrak{m}$, and C_0 denotes the fiber of C over $A'/\mathfrak{m} = k$. By [Kat96, Ex 5.6],

$$\Omega_{(X, \mathcal{M}_X)/k}^1 \simeq \mathcal{O}_X \otimes_{\mathbb{Z}} M.$$

Therefore,

$$\mathrm{Ext}^1(f_0^*\Omega_{(X, \mathcal{M}_X)/k}^1, \mathcal{O}_{C_0}) = H^1(\mathcal{O}_{C_0}^d) = 0$$

where the last equality holds because C_0 is a curve of arithmetic genus 0. This shows that $(\mathcal{K}_\Gamma(X), \mathcal{M}_{\mathcal{K}_\Gamma(X)})$ is log smooth.

To prove the claim about the dimension of $\mathcal{K}_\Gamma(X)$, note that

$$\dim \mathrm{Ext}^0(f_0^*\Omega_{(X, \mathcal{M}_X)/k}^1, \mathcal{O}_{C_0}) = \dim H^0(\mathcal{O}_{C_0}^d) = d,$$

and so π has relative dimension d . Since $\dim \mathfrak{M}_{0,n} = n-3$, we see $\dim \mathcal{K}_\Gamma(X) = d+n-3$. ♠

Let $\mathcal{K}_\Gamma^\circ(X)$ denote the non-degeneracy locus, that is, the locus of $\mathcal{K}_\Gamma(X)$ where the log structure $\mathcal{M}_{\mathcal{K}_\Gamma(X)}$ is trivial. By Proposition 2.1 and [Niz06, Prop 2.6], $\mathcal{K}_\Gamma^\circ(X)$ is an open dense subset of $\mathcal{K}_\Gamma(X)$. Consider the Kontsevich moduli space of stable maps $\mathfrak{M}_{0,n}(X, \beta)$. The forgetful map

$$\Phi : \mathcal{K}_\Gamma(X) \rightarrow \mathfrak{M}_{0,n}(X, \beta).$$

sending a stable log map to its underlying stable map induces a locally closed immersion

$$\mathcal{K}_\Gamma^\circ(X) \rightarrow \mathfrak{M}_{0,n}(X, \beta).$$

Let $\mathfrak{M}_\Gamma(X)$ be the closure of $\mathcal{K}_\Gamma^\circ(X)$ in $\mathfrak{M}_{0,n}(X, \beta)$. Then Φ factors through a morphism

$$\phi : \mathcal{K}_\Gamma(X) \rightarrow \mathfrak{M}_\Gamma(X).$$

Lemma 2.2. *ϕ is the normalization map.*

Proof. By [AC11, Corollary 3.10] and Proposition A.4, the morphism Φ is representable and finite, and so ϕ is as well. Since $(\mathcal{K}_\Gamma(X), \mathcal{M}_{\mathcal{K}_\Gamma(X)})$ is fs and log smooth over (k, \mathcal{O}_k^*) by Proposition 2.1, it follows that $\mathcal{K}_\Gamma(X)$ is normal. Since ϕ is an isomorphism over $\mathcal{K}_\Gamma^\circ(X)$, it is birational, and so by Zariski's Main Theorem, ϕ is the normalization map. ♠

For the rest of this section, we return to the setting and notation of the introduction, and let $\Gamma = \Gamma_0$. Just as $X//T_0$ (resp \mathfrak{M}) is constructed by taking the closure of T' in the Chow variety (resp the Kontsevich space), we can perform a similar construction with $\mathcal{K}_\Gamma(X)$. Namely, for $x \in T$, the stable map

$$f_x : \mathbb{P}^1 \rightarrow X$$

is naturally a stable log map. We therefore obtain a morphism $T' \rightarrow \mathcal{K}_\Gamma(X)$. Let \mathfrak{X}_Γ denote the closure of T' in $\mathcal{K}_\Gamma(X)$. The forgetful morphism Φ then induces a map

$$\phi' : \mathfrak{X}_\Gamma \rightarrow \mathfrak{M}.$$

Lemma 2.3. *\mathfrak{X}_Γ is an open substack of $\mathcal{K}_\Gamma(X)$, and so ϕ' is the normalization map.*

Proof. As in the proof of Lemma 2.2, ϕ' is representable and finite. If \mathfrak{X}_Γ is an open substack of $\mathcal{K}_\Gamma(X)$, it is then normal. Since ϕ' is an isomorphism over T' , Zariski's Main Theorem shows that it is the normalization map.

To show that \mathfrak{X}_Γ is open in $\mathcal{K}_\Gamma(X)$, it suffices to prove that $\mathcal{K}_\Gamma^\circ(X)$ and $\mathfrak{X}_\Gamma^\circ := \mathfrak{X}_\Gamma \cap \mathcal{K}_\Gamma^\circ(X)$ have the same dimension. Since T' is dense in \mathfrak{X}_Γ , we see that \mathfrak{X}_Γ has dimension $d-1$. On the other hand, the map

$$\pi : (\mathcal{K}_\Gamma(X), \mathcal{M}_{\mathcal{K}_\Gamma(X)}) \rightarrow (\mathfrak{M}_{0,2}, \mathcal{M}_{\mathfrak{M}_{0,2}})$$

in the proof of Proposition 2.1 induces a map

$$\mathcal{K}_\Gamma^\circ(X) \rightarrow \mathfrak{M}_{0,2}^\circ,$$

where $\mathfrak{M}_{0,2}^\circ$ denotes the open substack of $\mathfrak{M}_{0,2}$ with smooth fiber curves. By Proposition 2.1, we see that $\mathcal{K}_\Gamma^\circ(X)$ has dimension $d-1$. ♠

Since ϕ' is the normalization map, to prove Theorem 1.3, we must show $\mathfrak{X}_\Gamma = \mathcal{K}_\Gamma(X)$. Since \mathfrak{X}_Γ is an open and closed substack of $\mathcal{K}_\Gamma(X)$, the following proposition suffices.

Proposition 2.4. $\mathcal{K}_\Gamma(X)$ is irreducible.

Proof. It is enough to prove that $\mathcal{K}_\Gamma^\circ(X)$ is irreducible. Let $s \in \mathcal{K}_\Gamma^\circ(X)(k)$, and $f : \mathbb{P}^1 \rightarrow X$ be the stable log map corresponding to s . Note that the log structure of the boundary of X is everywhere non-trivial. Since the log structure is trivial at s , the image of f necessarily meets T . To prove that $\mathcal{K}_\Gamma^\circ(X)$ is irreducible, it is enough to show that we can act on f by an element of T to obtain a map isomorphic to f_1 from the introduction.

After acting on f by some element of T , we may assume that f sends $1 \in \mathbb{P}^1$ to $1 \in T \subset X$. Choose a maximal cone σ in the fan of X such that the associated affine open toric variety $U \subset X$ contains $f(0)$. Restricting f to U , we obtain a map $f' : V = \text{Spec } k[t] \rightarrow U$.

Let P be the monoid $\sigma^\vee \cap M$ and let e_1, \dots, e_ℓ be the irreducible elements of P . We see that for each i ,

$$f^*(e_i) = t^{c_i} a_i,$$

where c_i is the contact order prescribed by Γ and a_i is some element of $k[t]$. Note that if $\alpha \in k$ is a root of a_i , then the point $t = \alpha$ is mapped to the toric boundary; however, the contact order given by Γ implies that $t = 0$ is the only point in V which maps to the boundary. Hence, a_i must be a power of t . But if a_i is divisible by t , then the contact order of $t = 0$ along $e_i = 0$ is greater than c_i . Therefore, a_i must be a non-zero constant.

Now observe that the point $1 \in T \subset U$ is given by $e_i = 1$ for all i . Since $f(1) = 1$, the equation $f^*(e_i) = t^{c_i} a_i$ shows that $a_i = 1$. This shows that f is uniquely determined over U . Since f_1 also satisfies these constraints, we see that f and f_1 agree over U . Since f and f_1 are two maps from \mathbb{P}^1 to X which agree on a dense open subset of the source, they are equal. \spadesuit

3. TROPICAL CURVES ASSOCIATED TO STABLE LOG MAPS

The goal of this section is to prove Proposition 3.8. Following [NS06, GS11], we explain the connection between tropical curves and stable log maps to toric varieties.

3.1. Review of tropical curves. Let \overline{G} be the geometric realization of a weighted, connected finite graph with weight function ω . That is, \overline{G} is the CW complex associated to a finite connected graph with vertex set $\overline{G}^{[0]}$ and edge set $\overline{G}^{[1]}$, and

$$\omega : \overline{G}^{[1]} \rightarrow \mathbb{N}$$

is a function. Here we allow \overline{G} to have divalent vertices. Given an edge $l \in \overline{G}^{[1]}$, we denote its set of adjacent vertices by ∂l . If l is a loop, then we require $\omega(l) = 0$.

Let $G_\infty^{[0]} \subset \overline{G}^{[0]}$ be the set of one-valent vertices, and let

$$G := \overline{G} \setminus \overline{G}_\infty^{[0]}.$$

Let $G_\infty^{[1]}$ be the set of non-compact edges in G , which we refer to as *unbounded edges*. A *flag* of G is a pair (v, l) where l is an edge and $v \in \partial l$. We let FG be the set of flags of G , and for each vertex v , we let

$$FG(v) := \{(v, l) \in FG\}.$$

Let N be a lattice and $M = N^\vee$. We let $N_\mathbb{Q} := N \otimes_\mathbb{Z} \mathbb{Q}$ and $N_\mathbb{R} := N \otimes_\mathbb{Z} \mathbb{R}$.

Definition 3.2. A *parameterized tropical curve* in $N_\mathbb{Q}$ is a proper map $\varphi : G \rightarrow N_\mathbb{R}$ of topological spaces satisfying the following conditions:

- (1) For every edge l of G , the restriction $\varphi|_l$ acts as dilation by a factor $\omega(l)$ with image $\varphi(l)$ contained in an affine line with rational slope. If $\omega(l) = 0$, then $\varphi(l)$ is a point.
- (2) For every vertex v of G , we have $\varphi(v) \in N_\mathbb{Q}$.

- (3) For each $(v, l) \in FG(v)$, let $u_{v,l}$ be an primitive integral vector emanating from $\varphi(v)$ along the direction of $h(l)$. Then

$$\epsilon_v := \sum_{(v,l) \in FG(v)} \omega(l) u_{v,l} = 0,$$

which we refer to as the *balancing condition*.

An *isomorphism* of tropical curves $\varphi : G \rightarrow N_{\mathbb{R}}$ and $\varphi' : G' \rightarrow N_{\mathbb{R}}$ is a homeomorphism $\Phi : G \rightarrow G'$ compatible with the weights of the edges such that $\varphi = \varphi' \circ \Phi$.

A *tropical curve* is an isomorphism class of parameterized tropical curves.

3.3. Tropical curves from non-degenerate stable log maps. Let (X, \mathcal{M}_X) be a toric variety with its standard log structure, and let $T \subset X$ be its defining torus. We denote by N the lattice of one-parameter subgroups of T . Let $f : (C, \mathcal{M}_C) \rightarrow (X, \mathcal{M}_X)$ be a stable log map over (S, \mathcal{M}_S) with S a geometric point. Further assume that f is non-degenerate; that is, the log structure \mathcal{M}_S is trivial.

In this subsection, we show how to assign a tropical curve $\text{Trop}(f) : G \rightarrow N_{\mathbb{R}}$ to any such non-degenerate stable log map f . To begin, let G be the graph with a single vertex v , which we think of as being associated to the unique component of C , and with one unbounded edge for each marked point of C . We let $\text{Trop}(f)(v) = 0$.

Let l be an edge corresponding to a marked point p of C . If p has trivial contact orders, then we set $\omega(l) = 0$ and let $\text{Trop}(f)$ contract l to 0. Otherwise, the contact order is equivalent to giving a non-trivial map

$$c_l : \overline{\mathcal{M}}_{X,f(p)} \rightarrow \overline{\mathcal{M}}_{C,p} = \mathbb{N}.$$

Note that we have a surjective cospecialization map of groups

$$M := N^{\vee} \rightarrow \overline{\mathcal{M}}_{X,f(p)}^{gp}$$

corresponding to the specialization of the generic point of T to $f(p)$. Composing with c_l^{gp} , we obtain a map

$$\mu_l : M \rightarrow \mathbb{Z},$$

which defines an element $\mu_l \in N$. Let u_l be the primitive vector with slope given by $\mu_l \in N$. We define $\omega(l)$ to be the positive integer such that $\mu_l = \omega(l)u_l$, and define the image $\text{Trop}(f)(l)$ to be the unbounded ray emanating from 0 along the direction of u_l . This defines our desired map $\text{Trop}(f) : G \rightarrow N_{\mathbb{R}}$ up to reparameterization.

Proposition 3.4. $\text{Trop}(f) : G \rightarrow N_{\mathbb{R}}$ defines a tropical curve.

Proof. It remains to check that the balancing condition holds. That is, we must show $\epsilon_v = 0$. Note that every $m \in M$ defines a rational function on C and that the degree of the associated Cartier divisor is $0 = \epsilon_v(m)$. Therefore, $\epsilon_v \in N = M^{\vee}$ is 0. ♠

3.5. Tropical curves from stable log maps over the standard log point. Let (X, \mathcal{M}_X) be a toric variety with its standard log structure, and let $T \subset X$ be its defining torus. Fix discrete data $\Gamma = (g, \beta, n, \{c_i\})$ and let $f : (C, \mathcal{M}_C) \rightarrow (X, \mathcal{M}_X)$ be a stable log map with discrete data Γ over the standard log point (S, \mathcal{M}_S) ; that is, S is a geometric point and \mathcal{M}_S is the log structure associated to the map $\mathbb{N} \rightarrow \mathcal{O}_S$ sending 1 to 0. This is equivalent to giving a (not necessarily strict) log map

$$(S, \mathcal{M}_S) \rightarrow (\mathcal{K}_{\Gamma}(X), \mathcal{M}_{\mathcal{K}_{\Gamma}(X)}),$$

and the stable log map f is obtained by pulling back the universal stable log map over $(\mathcal{K}_{\Gamma}(X), \mathcal{M}_{\mathcal{K}_{\Gamma}(X)})$. In this subsection, we associate a tropical curve

$$\text{Trop}(f) : G \rightarrow N_{\mathbb{R}}$$

to f by modifying the construction given in [GS11, §1.3].

We define G to be the dual graph of C where we attach an unbounded edge for each marked point. Given a vertex v , let t be the generic point of the corresponding component of C . We therefore have a morphism

$$\overline{\mathcal{M}}_{X,f(t)} \rightarrow \overline{\mathcal{M}}_{C,t} = \mathbb{N}$$

of monoids. Taking the associated groups and composing with the cospecialization map $M \rightarrow \overline{\mathcal{M}}_{X,f(t)}^{gp}$ yields a map

$$\tau_v : M \rightarrow \mathbb{Z},$$

and hence a point in N . We define $\text{Trop}(f)(v) = \tau_v$.

Let l be an edge of G . If $\partial l = \{v, v'\}$ and $v \neq v'$, then we define the image of l under $\text{Trop}(f)$ to be the line segment joining τ_v and $\tau_{v'}$. In this case, $\tau_{v'} - \tau_v = e_l \mu_l$, where $e_l \in \overline{\mathcal{M}}_S = \mathbb{N}$ is the section which smooths the node corresponding to l , and μ_l is an element of N . We define $\omega(l)$ to be the positive integer such that $\mu_l = \omega(l)u_l$, where u_l is a primitive integral vector.

Suppose now that l is an unbounded edge corresponding to a marked point p . If p has trivial contact orders, then we set $\omega(l) = 0$ and let $\text{Trop}(f)$ contract l to τ_v , where $\partial l = \{v\}$. Otherwise, the contact orders of p define a non-trivial map

$$c_l : \overline{\mathcal{M}}_{X,f(p)} \rightarrow \overline{\mathcal{M}}_{C,p} = \mathbb{N} \oplus \overline{\mathcal{M}}_S \rightarrow \mathbb{N},$$

where the last map is the projection. Again taking the associated groups and composing with the cospecialization map $M \rightarrow \overline{\mathcal{M}}_{X,f(p)}^{gp}$, we obtain

$$\mu_l : M \rightarrow \mathbb{Z}.$$

We define $\omega(l)$ to be the positive integer such that $\mu_l = \omega(l)u_l$, where $u_l \in N$ is a primitive integral vector, and we let $\text{Trop}(f)(l)$ be the unbounded ray emanating from τ_v in the direction of u_l .

Proposition 3.6. $\text{Trop}(f) : G \rightarrow N_{\mathbb{R}}$ defines a tropical curve.

Proof. We must check that the balancing condition holds for each vertex v of G . As in the proof of Proposition 3.4, every $m \in M$ defines a rational function on the irreducible component of C corresponding to v . The degree of the associated Cartier divisor is $0 = \epsilon_v(m)$, and so $\epsilon_v = 0$, *c.f.* [GS11, Proposition 1.14]. ♠

Remark 3.7. Let R be the complete local ring of \mathbb{A}^1 at the origin, and let \mathcal{M}_R be the log structure on R induced by the standard log structure on \mathbb{A}^1 . Denote the closed and generic points of $\text{Spec } R$ by 0 and η , respectively. Suppose $h : (\mathcal{C}, \mathcal{M}_{\mathcal{C}}) \rightarrow (X, \mathcal{M}_X)$ is a stable log map over R with discrete data Γ such that $h_0 = f$. Note that h_{η} is a non-degenerate stable log map. For each marked section $p : \text{Spec } R \rightarrow \mathcal{C}$, let l_0 and l_{η} be the edges of the dual graphs of \mathcal{C}_0 and \mathcal{C}_{η} corresponding to the marked points p_0 and p_{η} , respectively. Consider the morphism

$$\overline{\mathcal{M}}_X|_{h(p)} \rightarrow \overline{\mathcal{M}}_{\mathcal{C}}|_p = \mathbb{N} \oplus \overline{\mathcal{M}}_R \rightarrow \mathbb{N},$$

where the last map is the projection. Taking associated groups and precomposing with the map $M \rightarrow \overline{\mathcal{M}}_X|_{h(p)}^{gp}$, we obtain a map $M \rightarrow \mathbb{Z}$ of constant sheaves on $\text{Spec } R$ whose special and generic fibers are μ_{l_0} and $\mu_{l_{\eta}}$. Hence, we see $\mu_{l_0} = \mu_{l_{\eta}}$.

The following result plays an important role in the proof of Theorem 1.1.

Proposition 3.8. *If the discrete data Γ is given by $g = 0$, $n = 2$, and $\beta \neq 0$, then $\text{Trop}(f)$ is an embedding whose image is a line. Moreover, C is a chain of \mathbb{P}^1 s and f does not contract any components of C .*

Proof. Since $\mathcal{K}_{\Gamma}(X)$ is log smooth by Proposition 2.1, there exists a stable log map $h : (\mathcal{C}, \mathcal{M}_{\mathcal{C}}) \rightarrow (X, \mathcal{M}_X)$ over (R, \mathcal{M}_R) as in Remark 3.7. Let $p, p' : \text{Spec } R \rightarrow \mathcal{C}$ be the two marked sections, and let $l_0, l'_0, l_{\eta}, l'_{\eta}$ be the corresponding edges of the dual graphs of \mathcal{C} and \mathcal{C}_{η} . Since $\beta \neq 0$, the two marked points p_{η} and p'_{η} of \mathcal{C}_{η} have non-trivial contact orders.

The balancing condition for $\text{Trop}(h_\eta)$ then shows $\mu_{l'_\eta} = -\mu_{l_\eta} \neq 0$. By Remark 3.7, we therefore have $\mu_{l'_0} = -\mu_{l_0} \neq 0$. In particular, $\text{Trop}(f)$ maps l_0 and l'_0 to unbounded rays.

We next show that if l is an edge of G , then $\text{Trop}(f)(l)$ is a point, or it is a line segment or ray parallel to μ_{l_0} . Suppose $\text{Trop}(f)(l)$ is not a point. If $\text{Trop}(f)(l)$ is unbounded, then l is l_0 or l'_0 , and so $\text{Trop}(f)(l)$ is parallel to μ_{l_0} . Otherwise, $\text{Trop}(f)(l)$ is a line segment and $\partial l = \{v, v_1\}$ with $v \neq v_1$. If $\text{Trop}(f)(l)$ is not parallel to μ_{l_0} , then the balancing condition shows that there is an edge $l_1 \neq l$ such that $v_1 \in \partial l_1$ and $\text{Trop}(f)(l_1)$ is not parallel to μ_{l_0} . Hence, l_1 is a line segment with endpoints v_1 and v_2 . Again, the balancing condition shows that there is an edge l_2 containing v_2 such that $\text{Trop}(f)(l_2)$ is a line segment which is not parallel to μ_{l_0} . Since C has genus 0, we see l , l_1 , and l_2 are distinct. Continuing in this manner, we produce an infinite sequence of distinct edges l_i of the dual graph of C . This is a contradiction.

Lastly, we show that every irreducible component A of C has exactly two special points. Hence, C is a chain of \mathbb{P}^1 s, f does not contract any component of C , and $\text{Trop}(f)(G)$ is a line parallel to μ_{l_0} . Suppose A is a component with at least three special points and let v be the vertex of G corresponding to A . Then $G \setminus v$ is a disjoint union of non-empty trees T_1, T_2, \dots, T_m with $m \geq 3$. Without loss of generality, T_1 only contains bounded edges. The argument in the preceding paragraph then shows that $\text{Trop}(f)$ maps every edge of T_1 to a single point. If C_1 denotes the subcurve of C corresponding to T_1 , then we see that every special point of C_1 has a trivial contact order, and so f contracts C_1 . Since T_1 is a tree, C_1 contains components with only two special points. This contradicts the stability of f . \spadesuit

4. THE CHOW QUOTIENT AS THE COARSE MODULI SPACE

Throughout this section, we let $\Gamma = \Gamma_0$ and $C(X)$ denote the Chow variety as in the introduction. Let K be the normalization of $X//T_0$. Note that there is a map

$$F : \mathcal{K}_\Gamma(X) \longrightarrow C(X)$$

sending a stable log map $f : (C, \mathcal{M}_C) \rightarrow (X, \mathcal{M}_X)$ to the image cycle $f_*[C]$. Since $\mathcal{K}_\Gamma(X)$ is irreducible by Theorem 1.3, F factors as

$$\mathcal{K}_\Gamma(X) \xrightarrow{F'} X//T_0 \xrightarrow{i} C(X),$$

where i is the natural inclusion. Since F is an isomorphism over T' and $\mathcal{K}_\Gamma(X)$ is normal, by Proposition 2.1, we obtain an induced morphism

$$G : \mathcal{K}_\Gamma(X) \longrightarrow K$$

To prove Theorem 1.1, we show

Proposition 4.1. *G is a coarse space morphism.*

Proof. Since both $\mathcal{K}_\Gamma(X)$ and K are normal and proper, and since G is an isomorphism over T' , by Zariski's Main Theorem, it suffices to show G is quasi-finite. To do so, it is enough to show F' is quasi-finite at the level of closed points. That is, we show that if $x \in X//T_0$ is a closed point and E_x denotes the corresponding cycle of X , then there are finitely many stable log maps whose image cycles are given by E_x . Let

$$E_x = \sum a_i Z_i,$$

where the a_i are positive integers and the Z_i are reduced irreducible closed subschemes of X . Let \tilde{Z}_i be the normalization of Z_i . Since E_x is of dimension 1, we have $\tilde{Z}_i \simeq \mathbb{P}^1$.

We claim that if $f : (C, \mathcal{M}_C) \rightarrow (X, \mathcal{M}_X)$ is a stable log map that defines a closed point of $\mathcal{K}_\Gamma(X)$ such that the image cycle of f is E_x , then f can only be ramified at the special points of C . Given this claim, F' is quasi-finite. Indeed, since Proposition 3.8 shows that

no component of C is contracted under f , the number of irreducible components of C is bounded by $\sum a_i$. For each irreducible component A of C , the map $f|_A$ factors as

$$A \longrightarrow \tilde{Z}_i \longrightarrow X$$

for some i . Since the first map $A \rightarrow \tilde{Z}_i$ can only be ramified at the two fixed special points, it is determined by the degree of $f|_A$. Thus, there are only finitely many choices for f .

It remains to prove the claim. By Proposition 2.4, $\mathcal{K}_\Gamma(X)$ is irreducible and T' is dense, so there exists a toric morphism $\mathbb{A}^1 \rightarrow \mathcal{K}_\Gamma(X)$ such that the fiber over $0 \in \mathbb{A}^1$ is our given stable log map $f : (C, \mathcal{M}_C) \rightarrow (X, \mathcal{M}_X)$ whose image cycle is E_x . Let R denote the complete local ring $\hat{\mathcal{O}}_{\mathbb{A}^1, 0}$ and let

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{h} & X \\ \downarrow & & \\ \text{Spec } R & & \end{array}$$

be the associated stable map. Let $\eta \in \text{Spec } R$ be the generic point.

We first handle the case when X is smooth. Let \mathcal{C}° be the open subset of \mathcal{C} obtained by removing the special points. Note that \mathcal{C}° is normal, and $h|_{\mathcal{C}^\circ}$ is quasi-finite by Proposition 3.8. By the purity of the branch locus theorem [AK71, p.461], if $h|_{\mathcal{C}^\circ}$ is ramified, then the ramification locus D is pure of codimension 1. Since $h|_{\mathcal{C}^\circ}$ is not everywhere ramified over the central fiber, D must intersect the generic fiber. However, $h|_{\mathcal{C}^\circ}$ is an embedding over the generic fiber, so we conclude that D is empty.

We now consider the case when X is singular. Let $p : \tilde{X} \rightarrow X$ be a toric resolution. We may replace R by a ramified extension, as this does not affect the set of closed points. Since the natural map $\mathcal{K}_\Gamma(\tilde{X}) \rightarrow \mathcal{K}_\Gamma(X)$ is proper, by the valuative criterion, we can assume we have a stable map $\tilde{h} : \tilde{\mathcal{C}} \rightarrow \tilde{X}$ and a commutative diagram

$$\begin{array}{ccc} \tilde{\mathcal{C}} & \xrightarrow{\tilde{h}} & \tilde{X} \\ q \downarrow & & \downarrow p \\ \mathcal{C} & \xrightarrow{h} & X \end{array}$$

over R . Here h is obtained by taking the stabilization of the prestable map $p \circ \tilde{h}$. The previous paragraph shows that \tilde{h} only ramifies at the special points. Since Proposition 3.8 shows that $\tilde{\mathcal{C}}$ and \mathcal{C} are both chains of \mathbb{P}^1 s, we see that h only ramifies at the special points as well. ♠

APPENDIX A: TORIC VARIETIES HAVE GENERALIZED DELIGNE-FALTINGS LOG STRUCTURES

The theory of moduli spaces of stable log maps $\mathcal{K}_\Gamma(Y, \mathcal{M}_Y)$ is developed in [Che11, AC11] and [GS11] for different classes of log schemes (Y, \mathcal{M}_Y) . In [Che11, AC11], Abramovich and the first author consider log schemes which are generalized Deligne-Faltings (see Definition A.1); in [GS11], Gross and Siebert consider log schemes which are quasi-generated Zariski. It is shown in [AC11, Prop 4.8] that when (Y, \mathcal{M}_Y) is both generalized Deligne-Faltings and quasi-generated Zariski, the Abramovich-Chen and Gross-Siebert constructions are identical. Gross-Siebert show that the standard log structure \mathcal{M}_X on a normal toric variety X is always quasi-generated Zariski. Here we show that if X is also projective, then \mathcal{M}_X is generalized Deligne-Faltings. Therefore, the two theories agree for projective normal toric varieties.

Definition A.1. A log structure \mathcal{M}_Y on a scheme Y is called *generalized Deligne-Faltings* if there exists a fine saturated sharp monoid P and a morphism $P \rightarrow \overline{\mathcal{M}}_Y$ which locally lifts to a chart $P \rightarrow \mathcal{M}_Y$.

Remark A.2. Given a fine saturated sharp monoid P , let $A_P = \operatorname{Spec} k[P]$ with its standard log structure \mathcal{M}_{A_P} . Then there is a natural action of $T_P := \operatorname{Spec} k[P^{gp}]$ on (A_P, \mathcal{M}_{A_P}) induced by the morphism $P \rightarrow P \oplus P^{gp}$ sending p to (p, p) . The log structure \mathcal{M}_{A_P} descends to yield a log structure $\mathcal{M}_{[A_P/T_P]}$ on the quotient stack $[A_P/T_P]$. By [Ols03, Rmk 5.15], a log scheme (Y, \mathcal{M}_Y) is generalized Deligne-Faltings if and only if there exists a strict morphism

$$(Y, \mathcal{M}_Y) \longrightarrow ([A_P/T_P], \mathcal{M}_{[A_P/T_P]})$$

for some fine saturated sharp monoid P .

Let X be a projective normal toric variety and let \mathcal{M}_X be its standard log structure. Let $Q \subset \mathbb{R}^n$ be a polytope associated to a sufficiently positive projective embedding of X . Placing Q at height 1 in $\mathbb{R}^n \times \mathbb{R}$ and letting P be the monoid of lattice points in the cone over Q , we have $X = \operatorname{Proj} k[P]$. Note that P is fine, saturated, and sharp. Let (A_P, \mathcal{M}_{A_P}) be as in Remark A.2, let U be the compliment of the closed subscheme of A_P defined by the irrelevant ideal of $k[P]$, and let $\mathcal{M}_U = \mathcal{M}_{A_P}|_U$. The function $\deg : P \rightarrow \mathbb{Z}$ sending an element to its height induces a \mathbb{G}_m -action on (A_P, \mathcal{M}_{A_P}) . Hence, \mathcal{M}_U descends to yield a log structure \mathcal{M}_P on X .

Lemma A.3. \mathcal{M}_P is generalized Deligne-Faltings.

Proof. We have a cartesian diagram

$$\begin{array}{ccc} (U, \mathcal{M}_U) & \longrightarrow & (A_P, \mathcal{M}_{A_P}) \\ \downarrow & & \downarrow \\ (X, \mathcal{M}_P) & \longrightarrow & ([A_P/\mathbb{G}_m], \mathcal{M}_{[A_P/\mathbb{G}_m]}) \end{array}$$

where all morphisms are strict and the vertical morphisms are smooth covers. Note that the \mathbb{G}_m -action on (A_P, \mathcal{M}_{A_P}) is induced from the morphism $\sigma : P \rightarrow P \oplus \mathbb{Z}$ defined by $p \mapsto (p, \deg p)$. Since σ factors as

$$P \longrightarrow P \oplus P^{gp} \longrightarrow P \oplus \mathbb{Z}$$

where the first map is $p \mapsto (p, p)$ and the second is $(p, \xi) \mapsto (p, \deg \xi)$, we see that there is a strict smooth cover

$$([A_P/\mathbb{G}_m], \mathcal{M}_{[A_P/\mathbb{G}_m]}) \longrightarrow ([A_P/T_P], \mathcal{M}_{[A_P/T_P]}).$$

Hence, Remark A.2 shows that \mathcal{M}_P is generalized Deligne-Faltings. ♠

Note that $\mathcal{M}_P|_T = \mathcal{O}_T^*$, where T is the torus of X . We therefore obtain a map

$$\psi : \mathcal{M}_P \longrightarrow j_*^{log} \mathcal{O}_T^* =: \mathcal{M}_X.$$

Proposition A.4. ψ is an isomorphism, and so (X, \mathcal{M}_X) is generalized Deligne-Faltings.

Proof. To show ψ is an isomorphism, it is enough to look Zariski locally on X . Note that X has an open cover by the $X_v := \operatorname{Spec} k[Q_v]$, where v is a vertex of the polytope Q and Q_v is the monoid of lattice points in the cone over $Q - v := \{q - v \mid q \in Q \subset \mathbb{R}^n\}$. Let P_v be the submonoid of P^{gp} generated by P and $-v$. Then we have a cartesian diagram

$$\begin{array}{ccc} A_{P_v} & \xrightarrow{i} & U \\ \pi \downarrow & & \downarrow \\ X_v & \longrightarrow & X \end{array}$$

where π is induced from the map $Q_v \rightarrow P_v$ embedding Q_v at height 0 in P_v , and where the composite of i and $U \rightarrow A_P$ is induced from the inclusion $P \rightarrow P_v$. Hence,

$$\mathcal{M}_{Q_v} = (\mathcal{M}_{P_v})^{\mathbb{G}_m}$$

and so ψ is an isomorphism over X_v . ♠

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